Linear Algebra and its Applications 435 (2011) 3297–3313

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

$\mathbb{Z}_2$-graded identities of the Grassmann algebra in positive characteristic

Lucio Centrone

Dipartimento di Matematica, Università degli Studi di Bari, Via Orabona 4, 70125 Bari, Italy

ARTICLE INFO

Article history:
Received 10 March 2011
Accepted 2 June 2011
Available online 12 July 2011

Submitted by P. Šemrl

AMS classification:
16R10
16W55
15A75

Keywords:
Polynomial identities
Grassmann algebra
Graded identities

ABSTRACT

Let F be an infinite field of characteristic $p > 2$ and let E be the Grassmann algebra generated by an infinite dimensional vector space V over F. In this paper, we describe the T_2-ideal of the $\mathbb{Z}_2$-graded polynomial identities of the Grassmann algebra E for any $\mathbb{Z}_2$-grading such that V is homogeneous in the grading. In particular, we give a description of the T_2-ideal of the graded identities of E in the case there is a finite number of homogeneous elements of the linear basis of E belonging to one of the homogenous components of E.

© 2011 Elsevier Inc. All rights reserved.

1. Introduction

The Grassmann algebra E arises naturally in many fields of physical and mathematical sciences. Mathematically speaking, it is the largest algebra that supports an alternating product on vectors, and can be easily defined in terms of other known objects such as tensors. The definition of the Grassmann algebra makes sense for spaces not just of geometric vectors, but of other vector-like objects such as vector fields or functions. The Grassmann algebra is also important in the theory of algebras with polynomial identities. More precisely, Kemer proved that any associative PI-algebra over a field F of characteristic zero is PI-equivalent to the Grassmann envelope of a finite dimensional associative superalgebra. Moreover, the matrix algebras $M_n(F)$, $M_n(E)$ over E and its subalgebras $M_{p,q}(E)$ $(p + q = n)$, generate the only non-trivial prime varieties. Over a field of characteristic 0, the latter algebras are the “building blocks” for the T-prime T-ideals (see [11]) and, when the base field is of positive

E-mail addresses: centrone@dm.uniba.it, luciocentrone@alice.it

0024-3795/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2011.06.008

characteristic, they turned out to be crucial too (see, for example, [12–14]). In [16], Latyshev dealt with the identities for the Grassmann algebra in characteristic 0 but his proof uses only characteristic different from 2. In [15], Krakowski and Regev proved that the polynomial $[x_1, x_2, x_3]$ forms a basis of the polynomial identities of E in the case that the ground field has characteristic 0. In [9], Giambruno and Koshlukov established that the last result is also true in the case the ground field is infinite of positive characteristic. In [17], Regev studied an analog of a problem of Procesi about matrices, i.e., if there are non-trivial polynomials over $\mathbb{Z}$ which become identities over $\mathbb{Z}_p$ for E. It turned out that such polynomials do not exist if $1 \in E$ but if $1 \notin E$ the answer is positive. In particular, x^p is a polynomial identity for the non-unitary Grassmann algebra E^*. In [4], Di Vincenzo gave a different proof of the result of Krakovsky and Regev and he also exhibited, for any k, finite bases of the identities of the Grassmann algebra of a k-dimensional vector space. The result of Di Vincenzo has been generalized in positive characteristic in [9], where the authors put in evidence how the fact that $p \leqslant k$ or $p > k$ affects the T-ideal of E.

In the light of this it seems an interesting problem to investigate more closely the structure of the graded polynomial identities of the Grassmann algebra. We recall some results about the graded identities of the Grassmann algebra in the case the ground field has characteristic 0. The structure of the $\mathbb{Z}_2$-graded identities of E with respect to its natural $\mathbb{Z}_2$-grading is well known (see [10]). Recently, Di Vincenzo and Da Silva give in [6] a complete description of the $\mathbb{Z}_2$-graded polynomial identities of E with respect to any $\mathbb{Z}_2$-grading such that the generating space is homogeneous, while da Silva computed the exact values of the $\mathbb{Z}_2$-graded codimensions of E in [3]. More generally, if E is $\mathbb{Z}_p$-graded, in [1] Anisimov made an algorithm to compute the exact value of the graded codimension of E for any $\mathbb{Z}_p$-grading of E, where p is a prime number. In his Ph.D. thesis [2] the author gives a description of the G-graded polynomial identities of E in the case G is a finite abelian group of odd order. Moreover he reduces the study of the T_G-ideal of E to some subgroups G' with $|G'| < |G|$ in the case G is a finite abelian group of any order.

In this paper, we shall study the analog of the problem studied by Di Vincenzo and Da Silva in [6]. In particular, we give a description of the T_2-ideal of E in the case the ground field is infinite of characteristic $p > 2$ for any $\mathbb{Z}_2$-grading such that the generating space is homogeneous of E. We shall see that the tool of the representation theory of the symmetric group is no longer useful in this case, then we are allowed to use the method of representations of the general linear group GL_m. Surprisingly, the generating polynomials of the T_2-ideals depend on the number k of homogeneous elements of the linear basis of E having homogeneous degree 0 or 1 and on the fact that $p \leqslant k$ or $p > k$. In all of these cases a crucial role is played by the graded identity z^p.

2. $\mathbb{Z}_2$-graded structures

In this section, we recall the definition of $\mathbb{Z}_2$-graded algebras (also called *superalgebras*) and $\mathbb{Z}_2$-graded polynomial identities. Moreover we recall the definition of Y-proper polynomials that are the main tool for the study of graded identities of graded algebras over infinite fields.

Definition 2.1. We say that A is a $\mathbb{Z}_2$-*graded algebra* or a *superalgebra* if $A = A^0 \oplus A^1$, where $A^0, A^1 \subseteq A$ are subspaces and $A^g A^h \subseteq A^{g+h}$ holds for $g, h \in \mathbb{Z}_2$. The subspace A^g is called the *homogeneous component* of A of degree g. We say that the elements $a \in A^g$ are homogeneous of degree g and we denote their degrees as $\|a\| = g$.

One defines $\mathbb{Z}_2$-graded subspaces of A, $\mathbb{Z}_2$-graded A-modules, $\mathbb{Z}_2$-graded homomorphisms and so on, in a standard way (see, for example, [8] or [5], for more details).

We denote by $F\langle Y, Z\rangle$ the free associative algebra generated by $X = Y \cup Z$, where $Y = \{y_1, y_2, \ldots\}$ and $Z = \{z_1, z_2, \ldots\}$ are two countable sets of disjoint variables. Given a map $\|\cdot\| : X \to \mathbb{Z}_2$, we can define a $\mathbb{Z}_2$-grading on $F\langle Y, Z\rangle$ if we set $\|w\| = \|x_{j_1}\| + \cdots + \|x_{j_n}\|$ for any monomial $w = x_{j_1} \cdots x_{j_n} \in F\langle Y, Z\rangle$. The homogeneous component $F\langle Y, Z\rangle^g \subseteq F\langle Y, Z\rangle$ is the subspace spanned by all monomials of degree g.

Definition 2.2. If A is a $\mathbb{Z}_2$-graded algebra, we denote by $T_{\mathbb{Z}_2}(A)$ the intersection of the kernels of all $\mathbb{Z}_2$-graded homomorphisms $F\langle Y, Z\rangle \to A$. Then $T_{\mathbb{Z}_2}(A)$ is a graded two-sided ideal of $F\langle Y, Z\rangle$ and its elements are called $\mathbb{Z}_2$-*graded polynomial identities* of the algebra A.

Note that $T_{\mathbb{Z}_2}(A)$ is stable under the action of any $\mathbb{Z}_2$-graded endomorphism of the algebra $F\langle Y, Z\rangle$. Any $\mathbb{Z}_2$-graded ideal of $F\langle Y, Z\rangle$ which verifies such property is said to be a $T_{\mathbb{Z}_2}$-ideal (or briefly a T_2-ideal). Clearly, any T_2-ideal I is the ideal of the $\mathbb{Z}_2$-graded polynomial identities of the graded algebra $F\langle Y, Z\rangle/I$. Note also that for a $\mathbb{Z}_2$-graded algebra A, the quotient algebra $F\langle Y, Z\rangle/T_2(A)$ is the relatively free algebra for the variety of graded algebras generated by A.

Let $H_h(F, d) \subseteq F\langle x_1, \dots, x_h\rangle$ be the vector space of homogeneous polynomials in the h variables $\{x_1, \dots, x_h | x_i \in \{y_i, z_i\}\}$ of degree d. If the ground field is infinite, a standard Vandermonde argument shows that $T_2(A)$ is generated, as a T_2-ideal, by the subspaces $T_2(A) \cap H_h(F, d)$. We define the factor space

$$H_h(A, d) = H_h(F, d)(A) := H_h(F, d)/(H_h(F, d) \cap T_2(A)),$$

then we shall denote with the symbol $c_d^{\mathbb{Z}_2}(A)$ its dimension that we shall call dth $\mathbb{Z}_2$-*graded homogeneous codimension of* A. A useful tool for the study of the structure of $H_h(A, d)$ is provided by the representation theory of the general linear group (see the book [7] of Drensky, for more details). Notice that $H_h(F, d)$ is a GL_d-module with respect to the natural left action and $H_h(F, d) \cap T_2(A)$ is a GL_d-submodule, hence $H_h(A, d)$ is a GL_d-module, as well. The study of the structure of $H_h(A, d)$ can be simplified considering some "specialized" subspaces of homogeneous polynomials. More precisely, we fix $s, t \in \mathbb{N}$ and we set

$$H_{s,t} := \mathrm{span}_F\langle w \textit{ monomials of } H_{s+t}(A, d) | y_1, \dots, y_s, z_1, \dots, z_t \textit{ occur in } w\rangle.$$

We observe that if $s + t = d$, the space $H_{s,t}$ is a $GL_s \times GL_t$-module, and the subspace $H_{s,t} \cap T_2(A)$ is a $GL_s \times GL_t$-submodule, too. Now we can form the factor space

$$H_{s,t}(A, d) := H_{s,t}/(H_{s,t} \cap T_2(A))$$

that is a $GL_s \times GL_t$-module, as well, and we shall denote with the symbol $c_{s,t}(A)$ its dimension.

Consider $[x_1, x_2] = x_1x_2 - x_2x_1$, the commutator of x_1 and x_2. Inductively one defines *left normed higher commutators* by

$$[x_1, \dots, x_{n-1}, x_n] = [[x_1, \dots, x_{n-1}], x_n], \quad n = 2, 3, \dots$$

In the sequel, we shall use the word commutators for the words left normed higher commutators.

Definition 2.3. A polynomial $f \in F\langle X\rangle$ is called a *proper polynomial* if it is a linear combination of products of commutators

$$f(x_1, \dots, x_n) = \sum \alpha_{i,\dots,j}[x_{i_1}, \dots, x_{i_p}] \cdots [x_{j_1}, \dots, x_{j_q}], \quad \alpha_{i,\dots,j} \in F.$$

Suppose now that we are dealing with $\mathbb{Z}_2$-graded algebras and consider the free algebra $F\langle Y, Z\rangle$. The Y-proper polynomials (see [8, Section 2]; [5, Section 2]) are the elements of the unitary F subalgebra $B(Y, Z)$ generated by the elements of Z and by all non-trivial commutators. Roughly speaking, a polynomial $f \in F\langle Y, Z\rangle$ is Y-proper if all the $y \in Y$ occurring in f appear in commutators only. Notice that if $f \in F\langle Z\rangle$, then f is Y-proper. It is well known (see, for instance, [5, Lemma 1, Section 2]) that all graded polynomial identities of a superalgebra A follow from the Y-proper ones. This means that the set $T_2(A) \cap B(Y, Z)$ generates the whole $T_2(A)$ as a T_2-ideal.

Let $\Gamma_h(F, d) \subseteq B(Y, Z)$ be the vector space of Y-proper homogeneous polynomials in the h variables $\{x_1, \dots, x_h | x_i \in \{y_i, z_i\}\}$ of degree d. We define the factor space

$$\Gamma_h(A, d) = \Gamma_h(F, d)(A) := \Gamma_h(F, d)/(\Gamma_h(F, d) \cap T_2(A)),$$

then we shall denote with the symbol $C_d^{\mathbb{Z}_2}(A)$ its dimension that we shall call dth $\mathbb{Z}_2$*-graded proper homogeneous codimension of* A. Notice that $\Gamma_h(F, d)$ is a GL_d-module with respect to the natural left action and $\Gamma_h(F, d) \cap T_2(A)$ is a GL_d-submodule, hence $\Gamma_h(A, d)$ is a GL_d-module, as well. The study of the structure of $\Gamma_h(A, d)$ can be simplified as in the previous case. We say that $w \in \Gamma_h(F, d)$ is a *Y-proper monomial*, or *monomial*, if it has the following form:

$$w = z_{i_1} \cdots z_{i_m}[z_{j_1}, \ldots, z_{j_2}] \cdots [z_{j_l}, \ldots, z_{j_t}][y_{r_1}, \ldots, y_{r_2}] \cdots [y_{r_a}, \ldots, y_{r_b}].$$

We fix $s, t \in \mathbb{N}$ and we set

$$\Gamma_{s,t} := \mathrm{span}_F \langle w \text{ monomials of } \Gamma_{s+t}(A, d) | y_1, \ldots, y_s, z_1, \ldots, z_t \text{ occur in } w \rangle.$$

We observe that if $s + t = d$, the space $\Gamma_{s,t}$ is a $GL_s \times GL_t$-module, and the subspace $\Gamma_{s,t} \cap T_2(A)$ is a $GL_s \times GL_t$-submodule, too. Now we can form the factor space

$$\Gamma_{s,t}(A, d) := \Gamma_{s,t}/(\Gamma_{s,t} \cap T_2(A))$$

that is a $GL_s \times GL_t$-module, as well, and we shall denote with the symbol $C_{s,t}(A)$ its dimension.

3. $\mathbb{Z}_2$-gradings on the Grassmann algebra *E*

All the fields we refer to are infinite of characteristic $p \neq 2$. Let V be a vector space over F of countable infinite dimension with basis $\{e_1, e_2, \ldots,\}$ and denote by V_k the subspace spanned by $e_1, e_2, \ldots, e_k$. The Grassmann algebra E of V is the associative algebra with F-basis consisting of 1 and all products of the form $e_{i_1}e_{i_2}\cdots e_{i_k}$ such that $i_1 < i_2 < \cdots < i_k = 1, 2, \ldots$ and with multiplication induced by $e_i^2 = 0$, $e_ie_j = -e_je_i$. Analogously, one defines the non-unitary Grassmann algebra E^* as the subalgebra of E generated by the products $e_{i_1}e_{i_2}\cdots e_{i_k}$ such that $i_1 < i_2 < \cdots < i_k$. Denote by E^0 the subspace of E spanned by 1 and by all basic elements of the form $e_{i_1}e_{i_2}\cdots e_{i_{2m}}$, where $m > 1$, and let E^1 be the subspace spanned by all elements of the form $e_{i_1}e_{i_2}\cdots e_{i_{2m+1}}$, where $m > 0$. Then E^0 is the center of E, and $ab = -ba$ for every $a, b \in E^1$.

We recall some well-known results about the Grassmann algebra that we shall use later on.

Theorem 3.1. *The infinite dimensional Grassmann algebra E over an infinite field satisfies the identity* $[x_1, x_2, x_3]$. *More precisely, the T-ideal of E, i.e.,* $T(E)$, *is generated by the polynomial* $[x_1, x_2, x_3]$.

We want to mention that in [16] Latyshev proved that if F has characteristic 0, then the T-ideal of $F\langle X\rangle$ generated by the polynomial $[x_1, x_2, x_3]$ is finitely generated as a T-ideal. The previous theorem has been proved by Krakovsky and Regev if the ground field has characteristic 0 (see [15]). In [9], Giambruno and Koshlukov give a complete proof of the theorem in the case the ground field is infinite with characteristic $p > 2$.

From now on we shall indicate with the symbol I the T-ideal $\langle [x_1, x_2, x_3]\rangle^T$ generated by the triple commutator. In particular, $I = T(E)$.

Lemma 3.2. *The polynomial* $[x_1, x_2][x_1, x_3]$ *belongs to I.*

Following word by word the work of Regev in [17], we have the following lemma:

Lemma 3.3. *If* $\mathrm{char}(F) = p > 0$, *then* E^* *satisfies the graded identity* x^p.

In this paper, we study the graded polynomial identities of E with respect any $\mathbb{Z}_2$-grading such that the generating vector space V is a homogeneous subspace. This is equivalent to consider a map

$$\varphi : V \to \mathbb{Z}_2$$

Now if $w = e_{i_1}e_{i_2}\cdots e_{in} \in E$ then the set $\mathrm{Supp}(w) := \{e_{i_1}, e_{i_2}, \ldots, e_{i_n}\}$ is the support of w and we define the $\mathbb{Z}_2$-grading of w by

$$\|e_{i_1}e_{i_2}\cdots e_{i_n}\| = \|e_{i_1}\| + \cdots + \|e_{i_n}\|.$$

If, for all $e_i \in B$, one has $\|e_i\| = |e_i| = 1 \in \mathbb{Z}_2$, then we obtain the natural $\mathbb{Z}_2$-grading on E, that is:

$$|e_{i_1}e_{i_2}\cdots e_{i_n}| := \begin{cases} 0 \text{ if } n \text{ is even} \\ 1 \text{ otherwise} \end{cases}$$

In this case, let E^0 be the homogeneous component of $\mathbb{Z}_2$-degree 0 and let E^1 be the component of degree 1. As we said above, $E^0 = Z(E)$ is the center of E and $ab + ba = 0$ for all $a, b \in E^1$. This means that E satisfies the following graded polynomial identities: $[y_1, y_2]$, $[y_1, z_1]$, $z_1z_2 + z_2z_1$. Now, let us consider the $\mathbb{Z}_2$-gradings on E induced by the maps $\|\cdot\|_{k*}$, $\|\cdot\|_\infty$, and $\|\cdot\|_k$, defined, respectively, by

$$\|e_i\|_{k*} = \begin{cases} 1 \text{ for } i = 1, \ldots, k \\ 0 \text{ otherwise,} \end{cases}$$

$$\|e_i\|_\infty = \begin{cases} 1 \text{ for } i \text{ odd} \\ 0 \text{ otherwise,} \end{cases}$$

$$\|e_i\|_k = \begin{cases} 0 \text{ for } i = 1, \ldots, k \\ 1 \text{ otherwise.} \end{cases}$$

4. Graded identities of E_{k*}

We start the study of the $\mathbb{Z}_2$-graded identities of the Grassmann algebra with the grading induced by the map $\|\cdot\|_{k^*}$. We shall see how the characteristic of the ground field affects the form of the T_2-ideal of the Grassmann algebra.

We use Lemma 3.3 to obtain the following result:

Proposition 4.1. *If* $\mathrm{char}(F) = p > 0$, *then* E_{k*} *satisfies the identity* z^p.

Proof. It is sufficient to observe that for any $w \in E^1_{k*}$ one has that $w \in E^*$, then the result follows in the light of Lemma 3.3. □

Lemma 4.2. *The monomial* $z_1\cdots z_{k+1}$ *is a* $\mathbb{Z}_2$*-graded identity of* E_{k^*}.

Proof. From the definition of E_{k^*} it follows that if $a \in E^1_{k^*}$ then a contains one among $e_1, \ldots, e_k$. Hence if $a_1, \ldots, a_{k+1} \in E^1_{k^*}$, the product $a_1\cdots a_{k+1} = \sum \alpha e_{i_1}\cdots e_{i_{k+1}}$ that is 0 because one among $e_1, \ldots, e_k$ repeats at least twice in each of the summands. □

The following statements will be crucial in what follows:

Lemma 4.3. *The polynomials* $t_{2n} = [y_1, y_2][y_3, y_4]\cdots[y_{2n-1}, y_{2n}]$ *are not* $\mathbb{Z}_2$*-graded identities of* E_{k*}, $n = 1, 2, \ldots$

Proof. It suffices to observe that $t_{2n}(e_{k+1}, \ldots, e_{k+2n}) = 2^n e_{k+1}\cdots e_{k+2n} \neq 0$. □

Lemma 4.4. *Consider the monomial $t = z_1^{r_1} \cdots z_l^{r_l}$ such that $\sum_{i=1}^{l} r_i \leqslant k$. Let $r = \max_i\{r_i\}$ and suppose that $p > r$. Then t is not a $\mathbb{Z}_2$-graded identity of E_{k*}.*

Proof. Consider the $\mathbb{Z}_2$-graded substitution $\varphi : F\langle Y, Z\rangle \to E$ such that:

$$\begin{aligned}
z_1 &\mapsto e_1 e_{k+1} + e_2 e_{k+2} + \cdots + e_{r_1} e_{k+r_1} \\
z_2 &\mapsto e_{r_1+1} e_{k+r_1+1} + \cdots + e_{r_1+r_2} e_{k+r_1+r_2} \\
&\vdots \\
z_i &\mapsto e_{r_1+\cdots+r_{i-1}+1} e_{k+r_1+\cdots+r_{i-1}+1} + \cdots + e_{r_1+\cdots+r_i} e_{k+r_1+\cdots+r_i} \\
&\vdots \\
z_l &\mapsto e_{r_1+\cdots+r_{l-1}+1} e_{k+r_1+\cdots+r_{l-1}+1} + \cdots + e_{r_1+\cdots+r_l} e_{k+r_1+\cdots+r_l}.
\end{aligned}$$

Due to the parity of $\varphi(z_i)$, we have that for any $i = 1, \ldots, l$

$$z_i^{r_i} = r_i! e_{r_1+\cdots+r_{i-1}+1} e_{k+r_1+\cdots+r_{i-1}+1} \cdots e_{r_1+\cdots+r_i} e_{k+r_1+\cdots+r_i}.$$

Finally we have that

$$\varphi\left(z_1^{r_1} \cdots z_l^{r_l}\right) = \pm \prod_{i=1}^{l} r_i! \cdot e_1 \cdots e_{r_1+\cdots+r_l} e_{k+1} \cdots e_{k+r_1+\cdots+r_l}.$$

Due to the fact that $p > r$, we have that p is not a factor of $\prod_{i=1}^{l} r_i!$, then the latter is different from 0 and we are done. □

Remark 4.5. Notice that the previous lemma is true even in the general case in which $\deg_{z_i} t = r_i$. In fact, the parity of the various $\varphi(z_i)$ gives us the possibility to use (in the computation) the same arguments used for $t = z_1^{r_1} \cdots z_l^{r_l}$. For example, if $t = z_1^{s_1} z_2^{r_2} z_1^{s_2} z_3^{r_3} \cdots z_l^{r_l}$, such that $s_1 + s_2 = r_1$, we consider the substitution φ of the theorem. We have that $\varphi(z_1)$ commutes with $\varphi(z_2)$, then $\varphi(t) = \varphi(z_1^{r_1} \cdots z_l^{r_l})$ and we are done. We can argue analogously for the general case.

We have all the necessary tools to state the main result of the section. In fact, we have that:

Theorem 4.6. *Let $p, k \in \mathbb{N}$, where p is prime. Over an infinite field F of characteristic $p \neq 2$ such that $p > k$, all $\mathbb{Z}_2$-graded polynomial identities of E_{k^*} are consequences of the graded identities:*

$$[x_1, x_2, x_3], \; z_1 \cdots z_{k+1}.$$

On the other side, if $p \leqslant k$, all $\mathbb{Z}_2$-graded polynomial identities of E_{k^*} are consequences of the graded identities:

$$[x_1, x_2, x_3], \; z_1 \cdots z_{k+1}, \quad z^p.$$

Proof. Suppose firstly that $p > k$. Let $f(x_1, \ldots, x_n) \in B(X) = B(Y, Z)$ be an identity of E and write $f = \sum \alpha_u u_1 \cdots u_k$, where the u_j's are Y-proper polynomials. Notice that the ground field is infinite, then we can assume f multihomogeneous. Due to the identity $[x_1, x_2, x_3]$ we may assume that all the commutators appearing in the u_j's are of the form $[x_a, x_b]$. The identity $z_1 \cdots z_{k+1}$ gives us the possibility to consider only at most k variables of $\mathbb{Z}_2$-degree 1 appearing in f. Hence f can be reduced to the form

$$\sum \alpha z_{i_1} \cdots z_{i_l} [y_1, y_2] \cdots [y_{s-1}, y_s][z_{j_1}, z_{j_2}] \cdots [z_{j_{m-1}}, z_{j_m}]$$

if s is even or

$$\sum \alpha z_{i_1} \cdots z_{i_l}[y_1, y_2] \cdots [y_{s-2}, y_{s-1}][y_s, z_{j_1}][z_{j_2}, z_{j_3}] \cdots [z_{j_{m-1}}, z_{j_m}]$$

if s is odd, where $l+m+s=n$, $l+m \leqslant k$ and the indices are ordered. We shall work only in the case s is even because the other case can be treated analogously. Let h be the number of different variables of degree 1 appearing in f and suppose that for any $i=1,\ldots,h$, $\deg_{z_i} f = r_i$. By Lemma 3.2 we have that f must be multilinear in the commutators, i.e.,

$$z_1^{d_1} \cdots z_h^{d_h}[y_1, y_2] \cdots [y_{s-1}, y_s][z_{j_1}, z_{j_2}] \cdots [z_{j_{m'-1}}, z_{j_{m'}}],$$

where $d_i \in \{r_i, r_i - 1\}$, $m' \leqslant m$ and the indices are ordered. Suppose on the contrary that the previous polynomials are not linearly independent modulo $T_{\mathbb{Z}_2}(E)$, then there exist non-zero coefficients such that

$$\sum \alpha z_1^{d_1} \cdots z_h^{d_h}[y_1, y_2] \cdots [y_{s-1}, y_s][z_{j_1}, z_{j_2}] \cdots [z_{j_{m'-1}}, z_{j'_m}] \in T_{\mathbb{Z}_2}(E).$$

Consider now the following substitution

$$\varphi : F\langle Y, Z\rangle \to E$$

such that:

$$\begin{aligned}
y_i &\mapsto e_{k+i} \\
z_1 &\mapsto e_1 e_{k+s+1} + e_2 e_{k+s+2} + \cdots + e_{r_1} e_{k+s+r_1} \\
z_2 &\mapsto e_{r_1+1} e_{k+s+r_1+1} + e_{r_1+2} e_{k+s+r_1+2} + \cdots + e_{r_1+r_2} e_{k+s+r_1+r_2} . \\
&\vdots \\
z_h &\mapsto e_{\sum_{j=1}^{h-1} r_j+1} e_{k+s+\sum_{j=1}^{h-1} r_j+1} + \cdots + e_{\sum_{j=1}^{h} r_j} e_{k+s+\sum_{j=1}^{h} r_j}
\end{aligned}$$

In the light of Lemmas 4.3 and 4.4, we have that the polynomial

$$f_1 = z_1^{r_1} \cdots z_h^{r_h}[y_1, y_2] \cdots [y_{s-1}, y_s]$$

is the only non-vanishing summand of f under the valuation φ. Then $\varphi(f) = 0$ if and only if the coefficient of f_1 is 0. Suppose now that the coefficient of f_1 is 0 and consider the following substitution

$$\psi_{a,b} : F\langle X\rangle \to E$$

such that:

$$\begin{aligned}
y_i &\mapsto e_{k+i} \\
z_1 &\mapsto e_1 e_{k+s+1} + e_2 e_{k+s+2} + \cdots + e_{r_1} e_{k+s+r_1} \\
z_2 &\mapsto e_{r_1+1} e_{k+s+r_1+1} + e_{r_1+2} e_{k+s+r_1+2} + \cdots + e_{r_1+r_2} e_{k+s+r_1+r_2} \\
&\vdots \\
z_a &\mapsto e_{\sum_{i=1}^{a-1} r_i+1} + e_{\sum_{i=1}^{a-1} r_i+2} e_{k+s+\sum_{i=1}^{a-1} r_i+2} + \cdots + e_{\sum_{i=1}^{a} r_i} e_{k+s+\sum_{i=1}^{a}} . \\
&\vdots \\
z_b &\mapsto e_{\sum_{i=1}^{b-1} r_i+1} + e_{\sum_{i=1}^{b-1} r_i+2} e_{k+s+\sum_{i=1}^{b-1} r_i+2} + \cdots + e_{\sum_{i=1}^{b} r_i} e_{k+s+\sum_{i=1}^{b}} \\
&\vdots \\
z_h &\mapsto e_{\sum_{j=1}^{h-1} r_j+1} e_{k+s+\sum_{j=1}^{h-1} r_j+1} + \cdots + e_{\sum_{j=1}^{h} r_j} e_{k+s+\sum_{j=1}^{h} r_j}
\end{aligned}$$

By Lemmas 4.3, 4.4 and their proofs we have that the evaluation under $\psi_{a,b}$ of the polynomial

$$f_{a,b} = z_1^{r_1} \cdots z_a^{r_a-1} \cdots z_b^{r_b-1} \cdots z_h^{r_h}[y_1, y_2] \cdots [y_{s-1}, y_s][z_a, z_b]$$

is

$$\pm 2^{s+1} \cdot \prod_{i \in \{1,\ldots,\widehat{a},\ldots,\widehat{b},\ldots,\}} r_i! \cdot (r_a - 1)!(r_b - 1)! \cdot e_1 \cdots e_{\sum_{i=1}^{h} r_i} e_{k+1} \cdots e_{k+s}$$

and that $f_{a,b}$ is the only non-vanishing summand of f under the valuation $\psi_{a,b}$. Then $\psi_{a,b}(f) = 0$ if and only if the coefficients of $f_{a,b}$ are 0. Arguing in the same way we are able to find substitutions ϕ such that $\phi(f) = 0$ if and only if the coefficient of each of the summands of f is 0. This easily shows the contradiction and we are done. If $p \leqslant k$, E_{k*} satisfies the graded identity z^p, then f can be reduced to the form

$$z_1^{d_1} \cdots z_h^{d_h}[y_1, y_2] \cdots [y_{s-1}, y_s][z_{j_1}, z_{j_2}] \cdots [z_{j_{m'-1}}, z_{j_{m'}}],$$

where $d_i \in \{r_i, r_i - 1\}$, $m' \leqslant m$, $d_i < p$ and the indices are ordered. We follow now word by word the proof for the case $p > k$ taking into account that $d_i < p$ make us able to use again Lemma 4.4. $\square$

5. The natural grading and E_∞

We shall exploit now the graded identities of the Grassmann algebra endowed with its natural grading and with the grading induced by the map $\|\cdot\|_\infty$. In what follows we shall use the so called *Jordan product* notation, i.e.,

$$x_1 \circ x_2 := x_1x_2 + x_2x_1$$

We observe that the argument used in Lemma 4.4 and Proposition 4.1 apply also in the case E is endowed with its natural grading and with the grading induced by the map $\|\cdot\|_\infty$, i.e., E_∞. In fact, we have that:

Lemma 5.1. *Consider the monomial $t = z_1^{r_1} \cdots z_l^{r_l}$ and let $r = \max_i\{r_i\}$. If $p > r$, then t is neither a $\mathbb{Z}_2$-graded identity of E nor of E_∞.*

The previous result provides us a useful argument to establish the following theorem:

Theorem 5.2. *Over an infinite field F of characteristic $p > 2$ all $\mathbb{Z}_2$-graded polynomial identities of E endowed with the natural grading are consequences of the graded identities:*

$$[y_1, y_2], \quad [y_1, z_1], \quad z_1 \circ z_2, \quad z^p.$$

Proof. The fact that the polynomials

$$[y_1, y_2], \quad [y_1, z_1], \quad z_1 \circ z_2$$

are graded identities of E is well known. Proposition 4.1 gives that z^p is a graded identity of E. Since $1 \in E^0$ it is sufficient to prove that all Y-proper identities of E follow from the above identities. Let $f(x_1, \ldots, x_n) \in B(Y, Z)$ be an identity of E and write $f = \sum \alpha_u u_1 \cdots u_k$, where the u_j's are Y-proper polynomials. Notice that the ground field is infinite, then we can assume f multihomogeneous. Due to the identities $[y_1, y_2]$ and $[y_1, z_1]$ we may assume that all the commutators appearing in the u_j's are of the form $[x_a, x_b]$. In the light of the graded identities $[y_1, y_2]$ and $[y_1, z_1]$ we have also that f does not contain variables of $\mathbb{Z}_2$-degree 0. The graded identities $z_1 \circ z_2$, z^p and the fact that f is multihomogeneous yield that f can be reduced to the form

$$f(x_1, \ldots, x_n) = \alpha z_1^{r_1} \cdots z_s^{r_s},$$

where $\alpha \in F$ and $r_i < p$. Hence if $\alpha \neq 0$ in F, by Lemma 5.1 we have that f is not a graded identity of E. Therefore $\alpha = 0$, and we are done. $\square$

We shall consider now the Grassmann algebra E_∞ endowed with the grading induced by the map $\|\cdot\|_\infty$. We have the following theorem:

Theorem 5.3. *Over an infinite field F of characteristic $p > 2$ all $\mathbb{Z}_2$-graded polynomial identities of E_∞ are consequences of the graded identities:*

$$[x_1, x_2, x_3], \quad z^p.$$

Proof. Let $f(x_1, \dots, x_n) \in B(Y, Z)$ be a multihomogeneous graded identity of E_∞ and write $f = \sum \alpha_u u_1 \cdots u_k$, where the u_j's are Y-proper polynomials. With the same notations of Theorem 4.6, it follows that f can be reduced to the form

$$\sum \alpha z_1^{d_1} \cdots z_h^{d_h} [y_1, y_2] \cdots [y_{s-1}, y_s][z_{j_1}, z_{j_2}] \cdots \left[z_{j_{m'-1}}, z_{j'_m}\right]$$

if s is even or

$$\sum \alpha z_1^{d_1} \cdots z_h^{d_h} [y_1, y_2] \cdots [y_{s-2}, y_{s-1}][y_s, z_{j_1}][z_{j_2}, z_{j_3}] \cdots \left[z_{j_{m'-1}}, z_{j'_m}\right]$$

if s is odd, where the indices are ordered. We shall work only in the case s is even because the other case can be treated analogously. Suppose by contradiction that the previous polynomials are not linearly independent modulo $T_{\mathbb{Z}_2}(E_\infty)$, then exist non-zero coefficients such that

$$\sum \alpha z_1^{d_1} \cdots z_h^{d_h} [y_1, y_2] \cdots [y_{s-1}, y_s][z_{j_1}, z_{j_2}] \cdots \left[z_{j_{m'-1}}, z_{j'_m}\right] \in T_{\mathbb{Z}_2}(E_\infty).$$

We follow now word by word the arguments used in Theorem 4.6. In particular, we consider the following substitution

$$\varphi : F\langle Y, Z\rangle \to E$$

such that:

$$\begin{gathered} y_i \mapsto e_{2i} \\ z_1 \mapsto e_1 e_{2(s+1)} + e_3 e_{2(s+2)} + \cdots + e_{2r_1-1} e_{2(s+r_1)} \\ z_2 \mapsto e_{2(r_1+1)-1} e_{2(s+r_1+1)} + e_{2(r_1+2)-1} e_{2(s+r_1+2)} + \cdots + e_{2(r_1+r_2)-1} e_{2(s+r_1+r_2)} . \\ \vdots \\ z_h \mapsto e_{2\left(\sum_{j=1}^{h-1} r_j+1\right)-1} e_{2\left(s+\sum_{j=1}^{h-1} r_j+1\right)} + \cdots + e_{2\sum_{j=1}^{h} r_j-1} e_{2\left(s+\sum_{j=1}^{h} r_j\right)} \end{gathered}$$

In the light of Lemmas 4.3 and 4.4, we have that the polynomial

$$f_1 = z_1^{r_1} \cdots z_h^{r_h} [y_1, y_2] \cdots [y_{s-1}, y_s]$$

is the only non-vanishing summand of f under the substitution φ. Then $\varphi(f) = 0$ if and only if the coefficient of f_1 is 0. Arguing in the same way we are able to find substitutions ϕ such that $\phi(f) = 0$ if and only if the coefficient of each of the summands of f is 0. This easily shows the contradiction and we are done. $\square$

6. Graded identities of E_k: the structure of $\Gamma_{0,m}(E_k)$ and $\Gamma_{1,m}(E_k)$

We start the study of the graded identities of E_k. Due to the fact that the elements of the homogeneous component E_k^0 have a lot of elements of odd length, the computations are very complicated. For this reason, it is desirable to study some special subspaces of homogeneous proper polynomials that give information about the structure of the T_2-ideal of E_k. We make use of the ideas developed by Di

Vincenzo and Da Silva in [6] adapted to the case of homogeneous polynomials instead of multilinear ones. We recall further that Proposition 4.1 gives us that the polynomial z^p is also a graded identity of E_k.

The following lemmas are well known:

Lemma 6.1. *If $l \equiv 0 (\mathrm{mod}\ 2)$, then $\Gamma_{l,m}$ is spanned modulo I by the polynomials*

$$z_{i_1}^{r_{i_1}} \cdots z_{i_s}^{r_{i_s}} [y_1, y_2] \cdots [y_{l-1}, y_l],$$

where $\sum_{j=1}^{s} r_{i_j} = m$. That is: for any $f \in \Gamma_{l,m}$ there exists $g \in \Gamma_{0,m}$ such that

$$f(y_1, \ldots, y_l, z_1, \ldots, z_s) \equiv g(z_1, \ldots, z_s)[y_1, y_2] \cdots [y_{l-1}, y_l] (\mathrm{mod}\ I).$$

Lemma 6.2. *If $l \equiv 1 (\mathrm{mod}\ 2)$ and $m \geqslant 1$, then $\Gamma_{l,m}$ is spanned modulo I by the polynomials*

$$z_{i_1}^{r_{i_1}} \cdots z_{i_s}^{r_{i_s}} [z, y_1] \cdots [y_{l-1}, y_l],$$

where $\sum_{j=1}^{s} r_{i_j} = m$. That is: for any $f \in \Gamma_{l,m}$ there exists $g \in \Gamma_{1,m}$ such that

$$f(y_1, \ldots, y_l, z_1, \ldots, z_s) \equiv g(z_1, \ldots, z_s, y_1)[y_2, y_3] \cdots [y_{l-1}, y_l] (\mathrm{mod}\ I).$$

In the even case we also have that:

Lemma 6.3. *If $l \equiv 0 (\mathrm{mod}\ 2)$ and $f \in \Gamma_{l,m}$, then*

(1) *if $l \geqslant k + 1$ then $f \in T_2(E_k)$,*
(2) *if $l \leqslant k$ then $f \in T_2(E_k) \Leftrightarrow g \in T_2(E_{k-l})$.*

In the odd case we have an analog of the previous lemma:

Lemma 6.4. *If $l \equiv 1 (\mathrm{mod}\ 2)$, $m \geqslant 1$ and $f \in \Gamma_{l,m}$, then*

(1) *if $l \geqslant k + 1$ then $f \in T_2(E_k)$,*
(2) *if $l \leqslant k$ then $f \in T_2(E_k) \Leftrightarrow g \in T_2(E_{k-l+1})$.*

The above lemmas lead us to study just $\Gamma_{0,m} \cap T_2(E_k)$ and $\Gamma_{1,m} \cap T_2(E_k)$ for all k.

We consider firstly $\Gamma_{0,m}$, then it is easy to see that $\Gamma_{0,m}$ is generated, modulo I, by the polynomials

$$z_{i_1}^{r_{i_1}} \cdots z_{i_s}^{r_{i_s}} [z_{j_1}, z_{j_2}] \cdots [z_{j_{t-1}}, z_{j_t}],$$

where t is even, $\sum_{j=1}^{s} r_{i_j} = m - t$ and the i_r's and the j_r's are, respectively, ordered. Analogously, $\Gamma_{0,m}$ is generated, modulo the T_2 ideal generated by I and z_p, by the polynomials

$$z_{i_1}^{r_{i_1}} \cdots z_{i_s}^{r_{i_s}} [z_{j_1}, z_{j_2}] \cdots [z_{j_{t-1}}, z_{j_t}],$$

where t is even, $\sum_{j=1}^{s} r_{i_j} = m - t$ and the i_r's and the j_r's are, respectively, ordered with the condition $r_{i_n} < p$. From now on we shall indicate with the symbol $I(p)$ the T_2-ideal generated by I and z^p.

Let $f = z_{i_1}^{r_{i_1}} \cdots z_{i_s}^{r_{i_s}} [z_{j_1}, z_{j_2}] \cdots [z_{j_{t-1}}, z_{j_t}]$ and consider the set

$$\mathcal{S} = \{\text{different homogeneous variables appearing in } f\} \subseteq \{z_1, \ldots, z_m\}.$$

If $h = |\mathcal{S}|$, then $\mathcal{S} = \{z_{i_1}, \ldots, z_{i_h}\}$. The polynomial f belongs now to the homogeneous component

$$\Gamma_{[h]} = (0, \ldots, 0, m_{i_1}, 0, \ldots, 0, m_{i_2}, 0, \ldots, 0, m_{i_h}, 0, \ldots, 0)$$

of $\Gamma_{0,m}$. With abuse of notation, sometimes we shall use the symbol $(m_1, \ldots, m_h)$ to indicate the homogeneous component

$$(0, \ldots, 0, m_{i_1}, 0, \ldots, 0, m_{i_2}, 0, \ldots, 0, m_{i_h}, 0, \ldots, 0).$$

Notice also that f is linear in the commutators. We consider now

$$T = \{j_1, \ldots, j_t\} \subseteq \mathcal{S}$$

and let us denote the previous polynomial by

$$f_T(z_{i_1}, \ldots, z_{i_h})$$

(respectively, $f_T(p)(z_{i_1}, \ldots, z_{i_h})$). Therefore any element of $\Gamma_{[h]}$ is, modulo I ($I(p)$), a linear combination of polynomials f_T ($f_T(p)$), that is

$$f \equiv \sum_T \alpha_T f_T (\mathrm{mod}\, I)$$

for some $\alpha_T \in F$.

Definition 6.5. For $m \geqslant 2$ let

$$g_m(z_{i_1}, \ldots, z_{i_h}) = \sum_{\substack{T \\ |\mathrm{T}|\ \mathrm{even}}} (-2)^{-\frac{|T|}{2}} f_T,$$

moreover put $g_1(z_1) = z_1$.

We recall the following facts without proofs (see [6] for more details):

Proposition 6.6. *The polynomial $g_{k+2}(z_1, \ldots, z_{k+2})$ is a $\mathbb{Z}_2$-graded polynomial identity of E_k.*

Definition 6.7. Let J_k be the T_2-ideal of the superalgebra $F\langle Y, Z\rangle$ generated by I and the polynomial $g_{k+2}(z_1, \ldots, z_{k+2})$ and $J_k(p)$ be the T_2-ideal of the superalgebra $F\langle Y, Z\rangle$ generated by $I(p)$ and $g_{k+2}(z_1, \ldots, z_{k+2})$.

Lemma 6.8. *In the free superalgebra $F\langle Y, Z\rangle$ we have:*

(1)

$$z_1 \cdots z_{k+2} \equiv \sum_{\substack{T \neq \emptyset \\ |\mathrm{T}|\ \mathrm{even}}} -(-2)^{-\frac{|T|}{2}} f_T (\mathrm{mod}\, J_k)(J_k(p))$$

(2)

$$z_2 \cdots z_{k+2}[z_1, z_{k+3}] \equiv \sum_{T'} \beta_{T'} f_{T'} (\mathrm{mod}\, J_k)(J_k(p))$$

for some $\beta_{T'} \in F$ and $T' \subseteq \{1, \ldots, k+3\}$, moreover if $|T'| = 2$, then $1 \notin T'$.

Remark 6.9. Under adequate substitution of the variables it is possible to obtain the analog of Lemma 6.8 for homogeneous polynomials. Suppose $k = 1$, and let F be a field of characteristic $p > 3$, then we have

$$z_1z_2z_3 \equiv \frac{1}{2} z_1[z_2, z_3] + \frac{1}{2} z_2[z_1, z_3] + \frac{1}{2} z_3[z_1, z_2] (\mathrm{mod}\, J_k).$$

We consider now all the possible multihomogeneous components (m_1, m_2, m_3) of total degree 3 and we consider the substitution φ such that $\varphi(z_i) = Z_j = z_j$ and for any $i = 1, 2, 3$ one has that $|Z_i| = m_i$.

Consider, for example, the multihomogeneous component $(2, 1, 0)$, then in the light of Lemma 6.8, we get the following equivalences modulo J_1:

- $z_1^2z_2 \equiv \frac{1}{2}z_1[z_1, z_2] + \frac{1}{2}z_1[z_1, z_2] = z_1[z_1, z_2]$,
- $z_1z_2z_1 \equiv -\frac{1}{2}z_1[z_1, z_2] + \frac{1}{2}z_1[z_1, z_2] = 0$,
- $z_2z_1^2 \equiv -\frac{1}{2}z_1[z_1, z_2] - \frac{1}{2}z_1[z_1, z_2] = -z_1[z_1, z_2]$.

We observe that if we consider the multihomogeneous component $(3, 0, 0)$, we get $z_1^3 \equiv 0$ and this shows that the identity z^p follows from $g_3(z_1, z_2, z_3)$.

We have now the following proposition:

Proposition 6.10. *For all $m \geqslant 1$, $\Gamma_{0,m}$ is spanned modulo J_k by*

$$\sum_{\Gamma_{[h]}} \sum_{s=m-(k+1)}^{h-1} \binom{h-1}{s}$$

polynomials. On the other hand, for all $m \geqslant 1$, $\Gamma_{0,m}$ is spanned modulo $J_k(p)$ by

$$\sum_{\Gamma_{[h]}} \sum_{s=m-(k+1)}^{h-1} \binom{h-1}{s}$$

polynomials, where the sum runs on all multihomogeneous components $(m_1, \ldots, m_h)$ such that $m_i \leqslant p$ for any $i = 1, \ldots, h$.

Proof. First of all, we observe that $\Gamma_{0,m}$ is generated by its homogeneous components. Let $\Gamma_{[h]}$ be one of those homogeneous components and, as we said above, $\Gamma_{[h]}$ is spanned modulo I by the polynomials $f_T = z_{i_1}^{r_{i_1}} \cdots z_{i_s}^{r_{i_s}}[z_{j_1}, z_{j_2}] \cdots [z_{j_{t-1}}, z_{j_t}]$, where $T = \{j_1, \ldots, j_t\}$, $|T| = t$ is even and the indices i_r's and j_r's are, respectively, ordered. By the previous lemma and the remark following it, if $m \geqslant k+2$, then modulo J_k, any polynomial f_T is a linear combination of polynomials f_S (belonging to the same homogeneous component). Moreover, the polynomials f_S are linear in the commutators. Then the total number of the f_S depends on the number of variables appearing in it, i.e., equals h. Let $s = |S|$, then if $m \equiv k+1 (\text{mod } 2)$ and $m - s = k+1$, we can assume $1 \notin S$. Let us denote by $d_{[h]} = d_{[h]}(k)$ the number of such polynomials f_S. We get the following values for $d_{[h]}$:

- if $m \equiv k \ (\text{mod } 2)$ then $d_{[h]} = \sum_{s=2, s \text{ even}, m-s \leqslant (k+1)}^{h} \binom{h}{s} = \sum_{s=m-(k+1)}^{h-1} \binom{h-1}{s}$
- if $m \equiv k+1 \ (\text{mod } 2)$ then $d_{[h]} = \binom{h-1}{m-(k+1)} + \sum_{s=2, s \text{ even}, m-s \leqslant k}^{h} \binom{h}{s} = \binom{h-1}{m-(k+1)} + \sum_{s=m-k}^{h-1} \binom{h-1}{s} = \sum_{s=m-(k+1)}^{h-1} \binom{h-1}{s}$.

Clearly if $m \leqslant k+1$ we get the desired result because in this case we have

$$\sum_{s=0, s \text{ even}}^{m} \binom{h}{s} = \sum_{s=0}^{m-1} \binom{h-1}{s}$$

spanning polynomials and we are done because $\binom{h-1}{s} = 0$ for all $s \geqslant h$.

Moreover, modulo $J_k(p)$, $\Gamma_{[h]}$ is spanned by the polynomials

$$f_T = z_{i_1}^{r_{i_1}} \cdots z_{i_s}^{r_{i_s}}[z_{j_1}, z_{j_2}] \cdots [z_{j_{t-1}}, z_{j_t}],$$

where $T = \{j_1, \ldots, j_t\}$, $|T| = t$ is even and the indices i_r's and j_r's are, respectively, ordered with the additional condition that $r_{i_1}, \ldots, r_{i_s}$ are bounded by $p - 1$. Now we may observe that the number of the generating polynomials depends again on the number of variables only and we are done. □

Consider now the following polynomials:

Definition 6.11. Let $\Gamma_{[h]}$ be a homogeneous component of $\Gamma_{0,m}$ and $l \leqslant h \leqslant m$. Let us define

$$t_l(z_{i_1}, \ldots, z_{i_h}) = \sum_\sigma S_l(z_{\sigma^{-1}(i_1)}, z_{i_2}, \ldots, z_{i_l}) z_{\sigma^{-1}(i_{l+1})} \cdots z_{\sigma^{-1}(i_h)},$$

$$t_l(p)(z_{i_1}, \ldots, z_{i_h}) = \sum_\sigma S_l(z_{\sigma^{-1}(i_1)}, z_{i_2}, \ldots, z_{i_l}) z_{\sigma^{-1}(i_{l+1})} \cdots z_{\sigma^{-1}(i_h)},$$

where $S_l(u_1, \ldots, u_l)$ denotes the standard polynomial of degree l and $\deg_{z_{i_s}} t_l(p) < p$.

Remark 6.12. The polynomials t_l, $t_l(p)$ correspond to the semi-standard Young tableau of the hook partition $(m - l + 1, 1^{l-1}) = (1 + b, 1^{m-b-1})$.

Lemma 6.13. *If $b = m - l \leqslant k$ and $p > k$, then t_l is not a graded identity of E_k. If $p \leqslant k$, then $t_l(p)$ is not a graded identity of E_k.*

Proof. Suppose firstly $p > k$. Since $b \leqslant k$, there exist at most b central elements of odd degree in E_k with pairwise disjoint supports. More precisely, let us consider the sets

$$\mathcal{S}_1 = \{\text{distinct variables appearing in the leg of the hook}\} = \{z_1, \ldots, z_{l-1}\},$$

$$\mathcal{S}_2 = \{\text{variables appearing only in the arm of the hook}\} = \{z_l, \ldots, z_h\},$$

then we consider the substitution $\varphi : F\langle Y, Z\rangle \to E$ such that:

$$\begin{aligned}
z_1 &\mapsto e_1 + e_2 e_{k+2} + \cdots + e_{n_1} e_{k+n_1} \\
z_2 &\mapsto e_{n_1+1} + e_{n_1+2} e_{k+n_1+2} + \cdots + e_{n_1+n_2} e_{k+n_1+n_2} \\
&\vdots \\
z_{l-1} &\mapsto e_{\sum_{j=1}^{l-2} n_j+1} + e_{\sum_{j=1}^{l-2} n_j+2} e_{k+\sum_{j=1}^{l-2} n_j+2} + \cdots + e_{\sum_{j=1}^{l-1} n_j} e_{k+\sum_{j=1}^{l-1} n_j} \\
z_l &\mapsto e_{\sum_{j=1}^{l-1} n_j+1} + e_{\sum_{j=1}^{l-1} n_j+2} e_{k+\sum_{j=1}^{l-1} n_j+2} + \cdots + e_{\sum_{j=1}^{l} n_j} e_{k+\sum_{j=1}^{l} n_j} \\
z_{l+1} &\mapsto e_{\sum_{j=1}^{l} n_j+1} e_{k+\sum_{j=1}^{l} n_j+1} + \cdots + e_{\sum_{j=1}^{l+1} n_j} e_{k+\sum_{j=1}^{l+1} n_j} \\
&\vdots \\
z_h &\mapsto e_{\sum_{j=1}^{h-1} n_j+1} e_{k+\sum_{j=1}^{h-1} n_j+1} + \cdots + e_{\sum_{j=1}^{h} n_j} e_{k+\sum_{j=1}^{h} n_j}.
\end{aligned}$$

First we assume that l is even, then $S_l(\varphi(z_i), \varphi(z_1), \ldots, \varphi(z_{l-1})) = 0$ for all $i \geqslant l + 1$ while $S_l(\varphi(z_l), \varphi(z_1), \ldots, \varphi(z_{l-1})) = l! e_1 e_{n_1+1} \cdots e_{\sum_{j=1}^{l-1} n_j+1}$. In this case we have that

$$\begin{aligned}
&t_l(\varphi(z_l), \varphi(z_1), \ldots, \varphi(z_{l-1}), \varphi(z_{l+1}), \ldots, \varphi(z_h)) \\
&= \pm l! \cdot \prod_{i=1}^{h} n_i'! \cdot e_1 e_{n_1+1} \cdots e_{\sum_{j=1}^{l-1} n_j+1} e_2 e_{k+2} \cdots e_{\sum_{j=1}^{l} n_j+1} \cdots e_{k+\sum_{j=1}^{h} n_j} \neq 0
\end{aligned}$$

because p does not appear as a factor of $l! \cdot \prod_{i=1}^{h} n_i'!$, where $n_i' \in \{n_i, n_i - 1\}$. If l is odd we have just to observe that

$$S_l(\varphi(z_i), \varphi(z_1), \ldots, \varphi(z_{l-1})) = \varphi(z_i)S_{l-1}(\varphi(z_1), \ldots, \varphi(z_{l-1}))$$
$$= \varphi(z_i)(l-1)!e_1 e_{n_1+1} \cdots e_{\sum_{j=1}^{l-1} n_j+1}$$

for all $i \geqslant l+1$. Then we can argue as in the previous case and we are done. If $p \leqslant k$ the proof is similar. In fact, we observe that for any $i = 1, \ldots, h$, we have $\deg_{z_i} = n_i < p$. Then the same substitution gets $l! \cdot \prod_{i=1}^{h} n_i'!$, where $n_i' < p$ for any $i = 1, \ldots, h$, and the latter is different from 0. □

Notice that for any subset $S_1 \subseteq \{z_1, \ldots, z_m\}$ of cardinality l, exist a finite set of polynomials, namely $t_l(S_1)$, $t_l(p)(S_1)$ such that the variables in $\mathcal{S}_1$ are the only ones appearing in the leg of the hook of the various $t_l \in t_l(S_1)$ or $t_l(p) \in t_l(p)(S_1)$. We consider now the polynomials

$$t_l^* = \sum_{\mathcal{S}_1 \subseteq \{z_1,\ldots,z_m\}} \sum_{t_l \in t_l(S_1)} t_l,$$

$$t_l(p)^* = \sum_{\mathcal{S}_1 \subseteq \{z_1,\ldots,z_m\}} \sum_{t_l(p) \in t_l(p)(S_1)} t_l(p).$$

We have the following lemma:

Lemma 6.14. *Let $m \geqslant 1$ and $p > k$, then t_l^* is not a graded identity of E_k. If $p \leqslant k$, then $t_l(p)^*$ is not a graded identity of E_k.*

Proof. Suppose firstly $p > k$. We fix the set of variables $S = \{z_1, \ldots, z_h\}$, then we consider the substitution φ such that φ maps the variables of S in the same elements as long as we presented in the proof of Lemma 6.13. For each of the remaining variables $\{z_{h+1}, \ldots, z_m\}$ we can substitute the homogeneous elements

$$z_{h+1} \mapsto e_{k+\sum_{j=1}^{h} n_j+1} + \cdots + e_{k+\sum_{j=1}^{h+1} n_j},$$

$$z_{h+2} \mapsto e_{k+\sum_{j=1}^{h+1} n_j+1} + \cdots + e_{k+\sum_{j=1}^{h+2} n_j}$$

and so on. With this in mind, it is easy to see that for each couple of distinct subsets of $\{z_1, \ldots, z_m\}$, φ gives out different elements of the basis of E with some coefficients. Hence we have to handle only with those hooks filled in with the same set of variables. Without loss of generality, we can work with the set of variables $\{z_1, \ldots, z_h\}$. Suppose that we fill the leg of the hook with a set of variables that is different with respect to the one used in the previous lemma. This implies that one of the central elements lies in the leg of the hook and the substitution vanishes. Then we restrict our problem to those hooks filled in with the same set of variables in the leg and fix two of them. We observe now that if $(n_1, \ldots, n_h)$ and $(m_1, \ldots, m_h)$ are the sequences of multiplicities of the variables $\{z_1, \ldots, z_h\}$ for those two hooks, exists $j \in \{1, \ldots, h\}$ such that $m_j > n_j$ and the substitution vanishes again. Hence the substitution does not vanish if and only if we have the hook filled in with variables of homogeneous degree $(n_1, \ldots, n_h)$ and we are done. If $p \leqslant k$ the proof is analogous. □

Remark 6.15. We recall from the standard theory of representation of the general linear group in characteristic 0 that if $m \in \mathbb{N}$, there is a one-to-one correspondence between irreducible GL_m-modules and partitions λ of m. In particular, if L^λ is an irreducible GL_m-module, then $L^\lambda = e_{T_\lambda} W$, where e_{T_λ} is a semi idempotent element of $F[S_m]$, $W = V^{\otimes m}$ and V is a vector space of dimension k. Moreover, $L^\lambda = \oplus_{\underline{d}} L^\lambda(\underline{d})$, where $\underline{d} = (d_1, \ldots, d_k)$ and $L^\lambda(\underline{d})$ is a semi-standard tableau of shape λ filled in with letters from 1 to k. Due to this fact, we can observe that each of the L^λ corresponds in a natural way to some homogeneous polynomial. If the ground field has characteristic $p > 2$, we can apply the same machinery but p has to be larger than the degree of the polynomials corresponding to the partitions. Otherwise our modules need not be irreducible, or we have no semisimplicity, or neither of these. In the light of this, Lemma 6.14 says that the polynomial t_l^* corresponds in a natural way to an irreducible GL_m-module.

Now we have the following proposition:

Proposition 6.16. *Let $m \geqslant 1$ and $p > k$, then $\Gamma_{0,m}(E_k) = \Gamma_{0,m}(J_k)$. If $p \leqslant k$, then $\Gamma_{0,m}(E_k) = \Gamma_{0,m}(J_k(p))$.*

Proof. Let $p > k$, then since $J_k \subseteq T_2(E_k)$, by Proposition 6.10 we have that

$$C_{0,m}(E_k) \leqslant \sum_{\Gamma_{[h]}} d_{[h]} = \sum_{\Gamma_{[h]}} \sum_{s=m-(k+1)}^{h-1} \binom{h-1}{s}.$$

On the other hand, by Remark 6.15, the irreducible module generated by the polynomial t_l^* determines a irreducible component of $\Gamma_{0,m}(E_k)$ of dimension $\binom{h-1}{l-1}$. Since these irreducible components are all distinct for all $l \geqslant m-k$, i.e., $l-1 \geqslant m-(k+1)$, we obtain

$$\sum_{\Gamma_{[h]}} \sum_{s=m-(k+1)}^{h-1} \binom{h-1}{s} \leqslant C_{0,m}(E_k)$$

and we are done. Let $p \leqslant k$ then since $J_k(p) \subseteq T_2(E_k)$, by Proposition 6.10 we have that

$$C_{0,m}(E_k) \leqslant \sum_{\Gamma_{[h]}} d_{[h]} = \sum_{\Gamma_{[h]}} \sum_{s=m-(k+1)}^{h-1} \binom{h-1}{s}.$$

On the other hand, by Remark 6.15, the irreducible module generated by the polynomial $t_l^*(p)$ determines a irreducible component of $\Gamma_{0,m}(E_k)$ of dimension $\binom{h-1}{l-1}$ and we are done. □

We consider now $\Gamma_{1,m}$. Similarly to the case of $\Gamma_{0,m}$, we define the homogeneous components of $\Gamma_{1,m}$. In this case the superalgebra E_k satisfies the following graded identities:

- $[g_{k+1}(z_1, \ldots, z_{k+1}), y]$
- $g_{k+1}(z_1, \ldots, z_{k+1})[z_{k+2}, y]$.

Definition 6.17. We set J_k' be the T_2-ideal generated by I and the polynomials $[g_{k+1}(z_1, \ldots, z_{k+1}), y]$ and $g_{k+1}(z_1, \ldots, z_{k+1})[z_{k+2}, y]$. On the other hand, we set $J_k'(p)$ be the T_2-ideal generated by $I(p)$ and the polynomials $[g_{k+1}(z_1, \ldots, z_{k+1}), y]$ and $g_{k+1}(z_1, \ldots, z_{k+1})[z_{k+2}, y]$.

As a consequence of these graded polynomial identities of E_k, using similar arguments to those given for the proof of Lemma 6.10, we obtain:

Proposition 6.18. *For all $m \geqslant 1$, $\Gamma_{1,m}$ is spanned modulo J_k' $(J_k(p)')$ by*

$$\sum_{\Gamma_{[h]}} \sum_{s=m-k}^{h-1} \binom{h-1}{s}$$

polynomials. On the other hand, for all $m \geqslant 1$, $\Gamma_{0,m}$ is spanned modulo $J_k(p)$ by

$$\sum_{\Gamma_{[h]}} \sum_{s=m-k}^{h-1} \binom{h-1}{s}$$

polynomials, where the sum runs on all multihomogeneous components $(m_1, \ldots, m_h)$ such that $m_i \leqslant p$ for any $i = 1, \ldots, h$.

Definition 6.19. Let $\Gamma_{[h]}$ be a homogeneous component of $\Gamma_{1,m}$ and $l \leqslant h \leqslant m$. Let us define

$$q_l(y, z_{i_1}, \ldots, z_{i_l}) = \sum_{\sigma} (-1)^{\sigma} [z_{\sigma(i_1)}, y] z_{\sigma(i_2)} \cdots z_{\sigma(i_l)}$$

$$t_l'(z_{i_1}, \ldots, z_{i_h}) = \sum_{\sigma} q_l(y, z_{\sigma^{-1}(i_1)}, z_{i_2}, \ldots, z_{i_l}) z_{\sigma^{-1}(i_{l+1})} \cdots z_{\sigma^{-1}(i_h)}$$

and

$$t_l'(p)(z_{i_1}, \ldots, z_{i_h}) = \sum_{\sigma} q_l(y, z_{\sigma^{-1}(i_1)}, z_{i_2}, \ldots, z_{i_l}) z_{\sigma^{-1}(i_{l+1})} \cdots z_{\sigma^{-1}(i_h)},$$

where $\deg_{z_{i_s}} t_l'(p) < p$.

Remark 6.20. The polynomial t_l' corresponds to the pair of semi-standard Young tableaux $(1, \mu_{m-l})$ where μ_{m-l} is the hook partition $(m - l + 1, 1^{l-1})$ of m.

Lemma 6.21. *If $b = m - l \leqslant k$ and $p > k$, then t_l' is not a graded identity of E_k. If $p \leqslant k$, then $t_l'(p)$ is not a graded identity of E_k.*

Notice that for any subset $S_1 \subseteq \{z_1, \ldots, z_m\}$ of cardinality l, exist a finite set of polynomials, namely $t_l'(S_1)$, $t_l'(S_1)(p)$, such that the variables in $\mathcal{S}_1$ are the only ones appearing in the leg of the hook of the various $t_l' \in t_l'(S_1)$ or $t_l'(p) \in t_l'(S_1)(p)$. We consider now the polynomials

$$t_l'^* = \sum_{\mathcal{S}_1 \subseteq \{z_1, \ldots, z_m\}} \sum_{t_l' \in t_l'(S_1)} t_l',$$

$$t_l'(p)^* = \sum_{\mathcal{S}_1 \subseteq \{z_1, \ldots, z_m\}} \sum_{t_l'(p) \in t_l'(S_1)(p)} t_l'(p).$$

We have the analog of Lemma 6.14:

Lemma 6.22. *Let $m \geqslant 1$ and $p > k$, then $t_l'^*$ is not a graded identity of E_k. If $p \leqslant k$, then $t_l'(p)^*$ is not a graded identity of E_k.*

Finally we have the following proposition:

Proposition 6.23. *Let $m \geqslant 1$ and $p > k$, then $\Gamma_{1,m}(E_k) = \Gamma_{1,m}(J_k')$. If $p \leqslant k$, then $\Gamma_{1,m}(E_k) = \Gamma_{1,m}(J_k'(p))$.*

7. Graded identities of E_k: the main result

We use the results of the previous section to obtain a description of the T_2-ideal of E_k. In what follows we use substantially the proof used by Di Vincenzo and Da Silva [6].

Theorem 7.1. *Let $T_2(E_k)$ be the T_2-ideal of E_k. Put $X = Y \cup Z$, then if $p > k$ $T_2(E_k)$ is generated by the set of the following polynomials:*

- $[x_1, x_2, x_3]$,
- $[y_1, y_2] \cdots [y_{k-1}, y_k][y_{k+1}, x]$ (if k is even)
- $[y_1, y_2] \cdots [y_k, y_{k+1}]$ (if k is odd)
- $g_{k-l+2}(z_1, \ldots, z_{k-l+2})[y_1, y_2] \cdots [y_{l-1}, y_l]$ (if $l \leqslant k$)
- $[g_{k-l+2}(z_1, \ldots, z_{k-l+2}), y_1][y_2, y_3] \cdots [y_{l-1}, y_l]$ (if $l \leqslant k$, l is odd)
- $g_{k-l+2}(z_1, \ldots, z_{k-l+2})[z, y_1][y_2, y_3] \cdots [y_{l-1}, y_l]$ (if $l \leqslant k$, l is odd).

If $p \leqslant k$ we have to add to the list above the identity

- z^p.

Proof. We can argue only for $p > k$ because the case $p \leqslant k$ can be treated similarly. Let P_k be the T_2-ideal of $F\langle Y, Z\rangle$ generated by the polynomials listed above accordingly to the parity of k. Clearly $I \subseteq P_k$ and it is easy to verify that $P_k \subseteq T_2(E_k)$. Hence it is enough to show that $\Gamma_{l,m} \cap T_2(E_k) \subseteq \Gamma_{l,m} \cap P_k$. Assume first that k and l are both even and let $f(y_1, \ldots, y_l, z_1, \ldots, z_t)$, where $\sum_{i=1}^{t} \deg_{z_i} f = m$ be a Y-proper graded polynomial identity of E_k. Since l is even then, by Lemma 6.1, there exists $g(z_1, \ldots, z_t) \in \Gamma_{0,m}$ such that $f \equiv g[y_1, y_2] \cdots [y_{l-1}, y_l] \pmod I$. Since $I \subseteq P_k$ the same conclusion holds modulo P_k. Since $[y_1, y_2] \cdots [y_{k-1}, y_k][y_{k+1}, x] \in P_k$, we can assume that $l \leqslant k$ and so by Lemma 6.3 we obtain that $g(z_1, \ldots, z_t)$ is a graded polynomial identity of the superalgebra E_{k-l}. Put $h = k - l$, then by Proposition 6.16 we have that g belongs to the T_2-ideal J_h. Hence, there exist certain monomials $v_i, w_i \in F\langle Y, Z\rangle$ and some graded endomorphisms φ_i of the free superalgebra such that $g \equiv \sum_i v_i \varphi_i(g_{h+2}) w_i \pmod I$. Since the commutators $[x_1, x_2]$ are central elements in the superalgebra $F\langle Y, Z\rangle / I$ we obtain:

$$f \equiv g[y_1, y_2] \cdots [y_{l-1}, y_l] \equiv \sum_i v_i \varphi_i(g_{h+2})[y_1, y_2] \cdots [y_{l-1}, y_l] w_i \pmod I.$$

Since $g_{h+2}(z_1, \ldots, z_{t'})$, $t' \leqslant h+2$, and $[y_1, y_2] \cdots [y_{l-1}, y_l]$ are polynomials in disjoint sets of indeterminate we can assume that $\varphi_i(y_j) = y_j$ for all i, j. This implies that each summand in the previous sum is an element of the T_2-ideal generated by the polynomial $g_{k-l+2}(z_1, \ldots, z_{k-l+2})[y_1, y_2] \cdots [y_{l-1}, y_l]$ which is one of the generators of P_k. Since $I \subseteq P_k$ we have the desired conclusion $f \in P_k$. Similar arguments hold for the remaining cases and we are done. $\square$

Acknowledgements

The author thanks Prof. O. M. Di Vincenzo for purposing him the problem. The author also thanks the referee for his/her useful comments.

References

[1] N. Anisimov, $\mathbb{Z}_p$-codimension of $\mathbb{Z}_p$-identities of Grassmann algebra, Comm. Algebra 29 (9) (2001) 4211–4230.
[2] L. Centrone, Polynomial identities of some minimal varieties with low PI-exponent, Ph.D. thesis, 2011.
[3] V.R.T. Da Silva, $\mathbb{Z}_2$-codimensions of the Grassmann algebra, Comm. Algebra 37 (9) (2009) 3342–3359.
[4] O.M. Di Vincenzo, A note on the identities of the Grassmann algebras, Boll. Unione Mat. Ital. Sez. A (7) 5 (3) (1991) 307–315.
[5] O.M. Di Vincenzo, V. Drensky, V. Nardozza, Subvarieties of the varieties of superalgebras generated by $M_{1,1}(E)$ or $M_2(K)$, Comm. Algebra 31 (1) (2003) 437–461.
[6] O.M. Di Vincenzo, V.R.T. Da Silva, On Z_2-graded polynomial identities of the Grassmann algebra, Linear Algebra Appl. 431 (1–2) (2009) 56–72.
[7] V. Drensky, Free Algebras and PI-Algebras. Graduate course in algebra, Springer-Verlag Singapore, 2000.
[8] V. Drensky, A. Giambruno, Cocharacters, codimensions and Hilbert series of the polynomial identities for 2×2 matrices with involution, Canad. J. Math. 46 (1994) 718–733.
[9] A. Giambruno, P. Koshlukov, On the identities of the Grassmann algebras in characteristic $p > 0$, Israel J. Math. 122 (1) (2001) 305–316.
[10] A. Giambruno, S. Mischenko, M.V. Zaicev, Polynomial identities on superalgebras and almost polynomial growth identities of Grassmann algebra, Comm. Algebra 29 (9) (2001) 3787–3800.
[11] A.R. Kemer, Ideals of Identities of Associative Algebras, Translations of Mathematical Monographs, vol. 87, American Mathematical Society, Providence, RI, 1991.
[12] A. Kemer, Remarks on the prime varieties, Israel J. Math. 96 (1995) 341–356.
[13] A. Kemer, Identities of algebras over a field of characteristic p, Rend. Circ. Mat. Palermo. (2) Suppl. 31 (1993) 133–148.
[14] A.R. Kemer, The standard identity in characteristic p. A conjecture of L.B. Volichenko, Israel. J. Math. 81 (1993) 343–355.
[15] D. Krakovski, A. Regev, The polynomial identities of the Grassmann algebra, Trans. Amer. Math. Soc. 181 (1973) 429–438.
[16] V. Latyshev, On the choice of basis in a T-ideal, Sibirsk. Mat. Zh. 4 (5) (1963) 1122–1126 (in Russian).
[17] A. Regev, Grassmann algebras over finite fields, Comm. Algebra 19 (6) (1991) 1829–1849.